Assignment 2. Solutions.

Problems. January 27.

For each of the following functions find i) a power series representation, ii) the radius convergence of the series.

1.

$$\frac{1}{(1-x)^3}$$

Solution.

We will use the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1.$$

Now differentiate this formula two times.

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1},$$

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}.$$

The radius of convergence does not change after differentiation. Therefore, the radius of convergence of the latter two series is also 1.

Dividing the latter equation by 2, we get

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}.$$

Or, if we change the index of summation,

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n.$$

2.

$$\frac{2x}{(1+3x^3)^3}$$

Solution. We will use the formula obtained above.

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n.$$

Replacing x by $-3x^3$ we get

$$\frac{1}{(1+3x^3)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} (-3x^3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n (n+2)(n+1)}{2} x^{3n}.$$

The latter series converges when $|-3x^3| < 1$. So, its radius of convergence is $1/\sqrt[3]{3}$. Multiplying the series by $2x$, we get

$$\frac{2x}{(1+3x^3)^3} = \sum_{n=0}^{\infty} (-1)^n 3^n (n+2)(n+1)\, x^{3n+1},$$

with radius of convergence $1/\sqrt[3]{3}$.

3.

$$\ln(1+x^2)$$

We again use the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

The latter has radius of convergence equal to 1.

After integration we get

$$-\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

The radius of convergence of this series is again 1.

In order to find the value of C, we set $x = 0$ in the latter equation.

$$-\ln(1) = C + \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1}.$$

So, $C = 0$. Hence,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Now replace x by $-x^2$ and get

$$\ln(1+x^2) = -\sum_{n=0}^{\infty} \frac{(-x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}.$$

The radius of convergence of this series is also one, since $|-x^2| < 1$ implies $|x| < 1$.

4.

$$x \arctan 2x^3$$

Solution.

We start with the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Now replace x by $-x^2$ to get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

The radius of convergence of this series is 1, since since $|-x^2| < 1$ implies $|x| < 1$.

Now integrate the formula we obtained above.

$$\arctan x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

The radius of convergence after integration remains the same, i.e. 1.

In order to find the value of C, we put $x = 0$ into the latter equality.

$$\arctan 0 = C + \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1}.$$

Therefore, $C = 0$, and we have

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Now replace x by $2x^3$. We get

$$\arctan(2x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{6n+3}}{2n+1}.$$

The radius of convergence can be determined as follows. $|2x^3| < 1$. So, $|x| < 1/\sqrt[3]{2}$. Thus, the radius of convergence is $1/\sqrt[3]{2}$.

Multiplying both sides in the above equality by x, we get

$$x \arctan(2x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{6n+4}}{2n+1}.$$

After this multiplication the radius of convergence remains the same, $1/\sqrt[3]{2}$.

Problems. January 29.

1. Find the Taylor series for $f(x) = 3^x$ at $x = 1$.

   Solution.

   The Taylor series for a function f centered at $x = a$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

   Therefore, we need to compute the derivatives of all orders of the given function at $x = 1$.

   $f(x) = 3^x$. Therefore,

$$f'(x) = \ln 3 \cdot 3^x,$$
$$f''(x) = (\ln 3)^2 3^x,$$
$$f'''(x) = (\ln 3)^3 3^x,$$

   and so on. Thus, the nth derivative (for all $n \geq 0$) equals

$$f^{(n)}(x) = (\ln 3)^n 3^x.$$

   Substituting $x = 1$, we get

$$f^{(n)}(1) = (\ln 3)^n 3.$$

   Thus, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{3(\ln 3)^n}{n!}(x-1)^n.$$

2. Find the Taylor series for $f(x) = \ln(2 + x)$ at $x = 1$.

   Solution.

   The required Taylor series is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n.$$

   Therefore, we need to compute the derivatives of all orders of the given function at $x = 1$.

$$f(x) = \ln(2 + x),$$

$$f'(x) = \frac{1}{2+x},$$

$$f''(x) = \frac{(-1)}{(2+x)^2},$$

$$f'''(x) = \frac{(-1)(-2)}{(2+x)^3},$$

$$f^{(4)}(x) = \frac{(-1)(-2)(-3)}{(2+x)^4},$$

$$\cdots$$

$$f^{(n)}(x) = \frac{(-1)(-2)(-3)\cdots(-(n-1))}{(2+x)^n} = \frac{(-1)^{n-1}(n-1)!}{(2+x)^n},$$

for all $n \geq 1$.

Thus,

$$f(1) = \ln 3$$

and

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!3^{-n}, \text{ for } n \geq 1.$$

The Taylor series for $f(x) = \ln(2+x)$ at $x = 1$ equals

$$\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!3^{-n}}{n!}(x-1)^n$$

$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n}(x-1)^n.$$

3. Find the Taylor polynomial of order 3 for $f(x) = \tan x$ at $x = 0$.

Solution.

The Taylor polynomial of order 3 for f at $x = a$ is given by

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

Thus we need to compute the derivatives up to order 3 of the function $f(x) = \tan x$ at $x = 0$.

We have

$$f'(x) = \sec^2 x,$$

$$f''(x) = 2\sec x \sec x \tan x = 2\sec^2 x \tan x,$$

$$f'''(x) = 4\sec x \sec x \tan x \tan x + 2\sec^2 x \sec^2 x = 4\sec^2 x \tan^2 x + 2\sec^4 x.$$

Now find their values at $x = 0$.

$$f(0) = 0,$$

$$f'(0) = 1,$$
$$f''(0) = 0,$$
$$f'''(0) = 2.$$

Therefore, the required Taylor polynomial equals

$$P_3(x) = x + \frac{2}{3!}x^3 = x + \frac{x^3}{3}.$$

4. Find the Taylor polynomial of order 4 for $f(x) = e^x \cos x$ at $x = 0$.

Solution.

The Taylor polynomial of order 4 for f at $x = a$ is given by

$$\begin{aligned} P_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4. \end{aligned}$$

Thus we need to compute the derivatives up to order 3 of the function $f(x) = e^x \cos x$ at $x = 0$.

$$f'(x) = e^x \cos x - e^x \sin x,$$

$$f''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x = -2e^x \sin x,$$
$$f'''(x) = -2e^x \sin x - 2e^x \cos x,$$
$$f^{(4)}(x) = -2e^x \sin x - 2e^x \cos x - 2e^x \cos x + 2e^x \sin x = -4e^x \cos x.$$

Now find their values at $x = 0$.

$$f(0) = 1,$$
$$f'(0) = 1,$$
$$f''(0) = 0,$$
$$f'''(0) = -2,$$

$$f^{(4)} = -4.$$

Therefore,

$$P_4(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4$$
$$= 1 + x - \frac{x^2}{3} - \frac{x^4}{6}.$$

Problems. February 1.

1. Let $f(x) = e^{x/2}$.

a) Find $P_4(x)$, the Taylor polynomial of order 4 for $f(x)$ at $x = 0$.

b) Use the remainder estimation theorem to estimate the error when $f(x)$ is replaced by $P_4(x)$ on the interval $[-1, 1]$.

Solution.

a) We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for all x.

Therefore,

$$e^{x/2} = \sum_{n=0}^{\infty} \frac{(x/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{2^n n!}.$$
$$= 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 3!} + \frac{x^4}{2^4 4!} + \cdots$$

Thus, taking the terms of order up to 4, we get the Taylor polynomial of order 4.

$$P_4(x) = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384}.$$

b) The Taylor estimation theorem says that

$$|f(x) - P_4(x)| \leq \frac{M}{5!}|x|^5,$$

where M is such a number that $|f^{(5)}(x)| \leq M$ on $[-1, 1]$.

Since $f^{(5)}(x) = \frac{e^{x/2}}{2^5}$, we have, for $x \in [-1, 1]$,

$$|f^{(5)}(x)| \leq \frac{e^{1/2}}{2^5} = \frac{\sqrt{e}}{32} = M.$$

Therefore,

$$|f(x) - P_4(x)| \le \frac{\sqrt{e}}{5! \cdot 32}|x|^5 \le \frac{\sqrt{e}}{5! \cdot 32},$$

since $|x| \le 1$.

Thus the error is at most $\dfrac{\sqrt{e}}{5! \cdot 32}$.

2. Find the derivative of order 15 of the function $f(x) = \arctan(x^3)$ at $x = 0$. (Hint: use the Maclaurin series).

Solution.

As we saw above (January 27, problem # 4),

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Therefore, replacing x by x^3, we get

$$f(x) = \arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1} = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \cdots .$$

Differentiating 15 times and putting $x = 0$, we get

$$f^{(15)}(0) = \frac{15 \cdot 14 \cdot 13 \cdots 2 \cdot 1}{5} = \frac{15!}{5}.$$

3. Find the binomial series for the function $f(x) = (1 - 2x)^{1/3}$. Write out the first four terms of the series.

Solution.

By the binomial theorem,

$$(1 + x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^n.$$

Replacing x by $-2x$, we get

$$(1 - 2x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} (-1)^n 2^n x^n.$$

Now compute the first four binomial coefficients.

$$\binom{1/3}{0} = 1, \binom{1/3}{1} = \frac{1}{3}, \binom{1/3}{2} = \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2} = -\frac{1}{9},$$

$$\binom{1/3}{3} = \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!} = \frac{5}{81}.$$

Hence,

$$(1-2x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} (-1)^n 2^n x^n = 1 - \frac{2}{3}x - \frac{4}{9}x^2 - \frac{40}{81}x^3 - \cdots$$

Problems. February 3.

1. Approximate the value of the integral with an error of magnitude less than 0.001.

$$\int_0^1 \frac{1-\cos x}{x^2} dx.$$

Solution. We know that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Then,

$$1 - \cos x = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{(2n)!},$$

$$\frac{1-\cos x}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n)!}.$$

After integration we get

$$\int_0^1 \frac{1-\cos x}{x^2} dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n)!(2n-1)}\Big|_0^1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!(2n-1)}$$

$$= \frac{1}{2} - \frac{1}{4! \cdot 3} + \frac{1}{6! \cdot 5} - \cdots = \frac{1}{2} - \frac{1}{72} + \frac{1}{3600} - \cdots$$

This is a geometric series. We need to find the first term that is less than 0.001. This is clearly 1/3600. Therefore, in order to approximate the series with the given accuracy, we need to take the sum of the terms that precede the latter one.

$$\frac{1}{2} - \frac{1}{72} = \frac{35}{72}.$$

2. Use series to evaluate the limit

$$\lim_{x\to 0} \frac{e^{2x} - (1 + 2x + 2x^2 + 2x^3)}{\sin x^3}.$$

Solution.

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

it follows that

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

On the other hand,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \cdots,$$

and so

$$\sin x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{6} + \cdots,$$

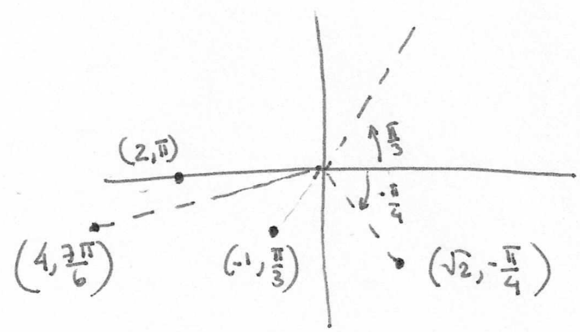
Therefore,

$$\lim_{x\to 0} \frac{e^{2x} - (1 + 2x + 2x^2 + 2x^3)}{\sin x^3}$$

$$= \lim_{x\to 0} \frac{(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots) - (1 + 2x + 2x^2 + 2x^3)}{x^3 - \frac{x^9}{6} + \cdots}$$

$$= \lim_{x\to 0} \frac{-\frac{2}{3}x^3 + \frac{2}{3}x^4 + \cdots}{x^3 - \frac{x^9}{6} + \cdots} = \lim_{x\to 0} \frac{-\frac{2}{3} + \frac{2}{3}x + \cdots}{1 - \frac{x^6}{6} + \cdots} = -\frac{2}{3}.$$

3. Plot the following points given in polar coordinates. Then find their Cartesian coordinates. a) $(\sqrt{2}, -\pi/4)$, b) $(-1, \pi/3)$, c) $(2, \pi)$, d) $(4, 7\pi/6)$.



a) $(\sqrt{2}, -\pi/4)$

$$x = r\cos\theta = \sqrt{2}\cos(-\pi/4) = 1,$$
$$y = r\sin\theta = \sqrt{2}\sin(-\pi/4) = -1.$$

Cartesian coordinates: $(1, -1)$.

b) $(-1, \pi/3)$

$$x = r\cos\theta = -1\cos(\pi/3) = -1/2,$$
$$y = r\sin\theta = -1\sin(\pi/3) = -\sqrt{3}/2.$$

Cartesian coordinates: $(-1/2, -\sqrt{3}/2)$.

c) $(2, \pi)$

$$x = r\cos\theta = 2\cos\pi = -2,$$
$$y = r\sin\theta = -1\sin\pi = 0.$$

Cartesian coordinates: $(-2, 0)$.

d) $(4, 7\pi/6)$

$$x = r\cos\theta = 4\cos(7\pi/6) = -2\sqrt{3},$$
$$y = r\sin\theta = 4\sin(7\pi/6) = -2.$$

Cartesian coordinates: $(-2\sqrt{3}, -2)$.

4. The following points are given in Cartesian coordinates. Find their polar coordinates. a) $(0, -1)$, b) $(-3, 3)$, c) $(2, -2\sqrt{3})$, d) $(\sqrt{3}, \sqrt{3})$.

a) $(0, -1)$

$$r = \sqrt{x^2 + y^2} = 1$$

$$\cos\theta = \frac{x}{r} = 0$$

$$\sin\theta = \frac{y}{r} = -1$$

So, $\theta = -\pi/2$.

Polar coordinates: $(1, -\pi/2)$.

b) $(-3, 3)$

$$r = \sqrt{x^2 + y^2} = 3\sqrt{2}$$

$$\cos\theta = \frac{x}{r} = -1/\sqrt{2}$$

$$\sin\theta = \frac{y}{r} = 1/\sqrt{2}$$

So, $\theta = 3\pi/4$.

Polar coordinates: $(3\sqrt{2}, 3\pi/4)$.

c) $(2, -2\sqrt{3})$

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4$$

$$\cos\theta = \frac{x}{r} = 1/2$$

$$\sin\theta = \frac{y}{r} = -\sqrt{3}/2$$

So, $\theta = -\pi/3$.

Polar coordinates: $(4, -\pi/3)$.

d) $(\sqrt{3}, \sqrt{3})$

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 3} = \sqrt{6}$$

$$\cos\theta = \frac{x}{r} = 1/\sqrt{2}$$

$$\sin\theta = \frac{y}{r} = 1/\sqrt{2}$$

So, $\theta = \pi/4$.

Polar coordinates: $(\sqrt{6}, \pi/4)$.

Problems. February 5.

1. Find a polar equation for the circle $(x-3)^2 + (y+4)^2 = 25$. Simplify your answer.

   Solution.

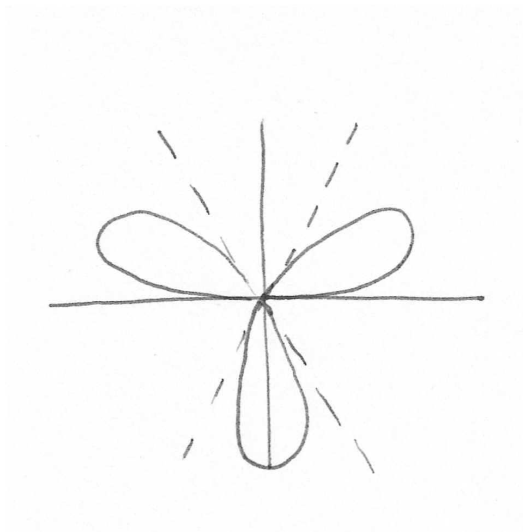
   Using
$$x = r\cos\theta, \quad y = r\sin\theta$$
   we get
$$(r\cos\theta - 3)^2 + (r\sin\theta + 4)^2 = 25,$$
$$r^2\cos^2\theta - 6r\cos\theta + 9 + r^2\sin^2\theta + 8r\sin\theta + 16 - 25 = 0,$$
$$r^2\cos^2\theta + r^2\sin^2\theta - 6r\cos\theta + 8r\sin\theta = 0,$$
$$r^2 - 6r\cos\theta + 8r\sin\theta = 0,$$
$$r - 6\cos\theta + 8\sin\theta = 0,$$
$$r = 6\cos\theta - 8\sin\theta.$$

2. Convert the equation $r = \tan\theta\sec\theta$ from polar coordinates to Cartesian coordinates. Identify the curve.

   Solution.
$$r = \frac{\sin\theta}{\cos^2\theta}$$
$$r\cos^2\theta = \sin\theta$$
$$r^2\cos^2\theta = r\sin\theta$$
$$x^2 = y.$$
   This is a parabola.

3. Sketch the curve $r = \sin 3\theta$.

   Solution. The points where the graph passes through the origin:
$$\sin 3\theta = 0.$$
   So, $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$.



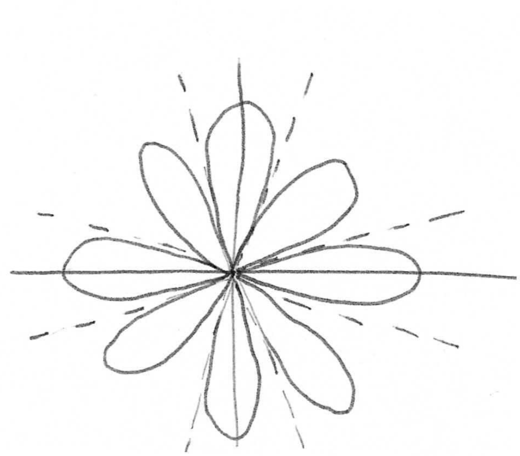
Three-leaved rose.

4. Sketch the curve $r = \cos 4\theta$.

   Solution. The points where the graph passes through the origin:

$$\cos 4\theta = 0.$$

   So, $\theta = \pi/8, 3\pi/8, 5\pi/8, 7\pi/8, 9\pi/8, 11\pi/8, 13\pi/8, 15\pi/8.$



Eight-leaved rose.

Problems. February 8.

1. Find the slope of the four-leaved rose $r = \cos 2\theta$ at the points where $\theta = 0, \pi/6, \pi/4$.

Solution.

Using the relation between polar and Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$x = \cos 2\theta \cos\theta,$$
$$y = \cos 2\theta \sin\theta.$$

Now differentiate with respect to θ.

$$\frac{dx}{d\theta} = -2\sin 2\theta \cos\theta - \cos 2\theta \sin\theta,$$

$$\frac{dy}{d\theta} = -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta.$$

The slope is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta}{-2\sin 2\theta \cos\theta - \cos 2\theta \sin\theta}$$

Now substitute the given values $\theta = 0, \pi/6, \pi/4$.

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \infty,$$

that is the tangent line is vertical.

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-2\sin(\pi/3)\sin(\pi/6) + \cos(\pi/3)\cos(\pi/6)}{-2\sin(\pi/3)\cos(\pi/6) - \cos(\pi/3)\sin(\pi/6)} = \frac{-2\cdot\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}}{-2\cdot\frac{3}{4} - \frac{1}{4}}$$

$$= \frac{-\frac{\sqrt{3}}{4}}{-\frac{7}{4}} = \frac{\sqrt{3}}{7}.$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{-2\sin(\pi/2)\sin(\pi/4) + \cos(\pi/2)\cos(\pi/4)}{-2\sin(\pi/2)\cos(\pi/4) - \cos(\pi/2)\sin(\pi/4)} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1.$$

2. Find the points of intersection of the curves $r^2 = \cos\theta$ and $r^2 = \sin\theta$.

   Solution.

   If we set the equations $r^2 = \cos\theta$ and $r^2 = \sin\theta$ equal to each other, we get

   $$\cos\theta = \sin\theta,$$
   $$\tan\theta = 1,$$
   $$\theta = \frac{\pi}{4}.$$

   Now solve

   $$r^2 = \cos\frac{\pi}{4} = 1/\sqrt{2}.$$
   $$r = \pm 1/\sqrt[4]{2}.$$

   So, we get $(1/\sqrt[4]{2}, \frac{\pi}{4})$, $(-1/\sqrt[4]{2}, \frac{\pi}{4})$ as points of intersection.

   Now notice that both curves are symmetric with respect to the x-axis. Indeed, if (r,θ) belongs to the curve $r^2 = \cos\theta$, then so does $(r,-\theta)$. Similarly, if (r,θ) belongs to the curve $r^2 = \sin\theta$, then so does $(-r, \pi-\theta)$.

   Since both curves are symmetric with respect to the x-axis, we get two more points of intersection: $(1/\sqrt[4]{2}, -\frac{\pi}{4})$, $(-1/\sqrt[4]{2}, -\frac{\pi}{4})$.

   Finally, notice that origin belongs to both curves. Thus, the fifth point of intersection is $(0,0)$.

3. Find the area inside one leaf of the three-leaved rose $r = \sin 3\theta$.

   Solution.

   In order to find the angles that define one leaf of the curve, we set

   $$\sin 3\theta = 0$$

   and get $\theta = 0$ and $\theta = \pi/3$.

   Therefore,

   $$A = \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta d\theta = \frac{1}{4}\int_0^{\pi/3}(1-\cos 6\theta)d\theta$$
   $$= \frac{1}{4}(\theta - \frac{1}{6}\sin 6\theta)\Big|_0^{\pi/3} = \frac{\pi}{12}.$$

Problems. February 10.

1. Sec. 10.7 # 8. Find the area of the region shared by the circles $r = 1$ and $r = 2\sin\theta$.

   Solution.

   First let us find the points of intersection of these two circles. Setting the two functions equal to each other, we get the equation

$$2\sin\theta = 1,$$

$$\sin\theta = \frac{1}{2},$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

   Since both circles are symmetric with respect to the y-axis, we will only consider that part of the region that lies in the first quadrant. This region consists of two parts:

   i) if $0 \le \theta \le \pi/6$, the region is bounded by the curve $r = 2\sin\theta$.

   ii) if $\pi/6 \le \theta \le \pi/2$, the region is bounded by the curve $r = 1$.

   So, one half of the area equals

$$\frac{1}{2}A = \frac{1}{2}\left(\int_0^{\pi/6}(2\sin\theta)^2 d\theta + \int_{\pi/6}^{\pi/2} 1 d\theta\right) = \frac{1}{2}\left(\int_0^{\pi/6} 2(1-\cos 2\theta)d\theta + \theta\Big|_{\pi/6}^{\pi/2}\right)$$

$$= \frac{1}{2}\left(2(\theta - \frac{1}{2}\sin 2\theta)\Big|_0^{\pi/6} + \theta\Big|_{\pi/6}^{\pi/2}\right) = \frac{1}{2}\left(\frac{\pi}{3} - \sin\frac{\pi}{3} + \frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

   So,

$$A = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

2. Sec. 10.7 # 12 Find the area of the region inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1+\cos\theta)$, $a > 0$.

   Solution.

   Let us find the points of intersection of the two curves.

$$3a\cos\theta = a(1+\cos\theta),$$

$$3\cos\theta = 1 + \cos\theta,$$

$$2\cos\theta = 1,$$

$$\theta = \pm\frac{\pi}{3}.$$

The region is bounded outside by the curve $r = 3a\cos\theta$ and inside by the curve $r = a(1+\cos\theta)$. So, using the symmetry with respect to the x-axis, we get

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/3}\left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right]d\theta,$$

$$= a^2\int_0^{\pi/3}\left[9\cos^2\theta - (1+2\cos\theta+\cos^2\theta)\right]d\theta$$

$$= a^2\int_0^{\pi/3}\left[8\cos^2\theta - 1 - 2\cos\theta\right]d\theta$$

using the double-angle formula,

$$= a^2\int_0^{\pi/3}[4+4\cos 2\theta - 1 - 2\cos\theta]\,d\theta$$

$$= a^2\,[3\theta + 2\sin 2\theta - 2\sin\theta]\Big|_0^{\pi/3} = a^2\left[\pi + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right]$$

$$= \pi a^2.$$

3. Sec. 10.7 # 22. Find the length of the curve $r = a\sin^2(\theta/2)$, $0 \le \theta \le \pi$, $a > 0$.

   Solution.

$$L = \int_0^{\pi}\sqrt{a^2\sin^4(\theta/2) + (2a\sin(\theta/2)\frac{1}{2}\cos(\theta/2))^2}d\theta$$

$$= a\int_0^{\pi}\sqrt{\sin^4(\theta/2)+\sin^2(\theta/2)\cos^2(\theta/2)}d\theta$$

$$= a\int_0^{\pi}\sin(\theta/2)\sqrt{\sin^2(\theta/2)+\cos^2(\theta/2)}d\theta$$

$$= a\int_0^{\pi}\sin(\theta/2)d\theta = -2a\cos(\theta/2)\Big|_0^{\pi} = 2a.$$

4. Sec. 10.7 # 30. Find the area of the surface generated by revolving the curve $r = \sqrt{2}e^{\theta/2}$, $0 \le \theta \le \pi/2$ about the x-axis.

   Solution.

$$
\begin{aligned}
S &= \int_0^{\pi/2} 2\pi\sqrt{2}e^{\theta/2}\sin\theta\sqrt{(\sqrt{2}e^{\theta/2})^2 + (\frac{\sqrt{2}}{2}e^{\theta/2})^2}d\theta \\
&= 2\sqrt{2}\pi\int_0^{\pi/2} e^{\theta/2}\sin\theta\sqrt{2e^{\theta} + \frac{1}{2}e^{\theta}}d\theta \\
&= 2\sqrt{2}\pi\int_0^{\pi/2} e^{\theta/2}\sin\theta\cdot e^{\theta/2}\sqrt{\frac{5}{2}}d\theta \\
&= 2\sqrt{5}\pi\int_0^{\pi/2} e^{\theta}\sin\theta\ d\theta
\end{aligned}
$$

Now let us separately compute the following integral

$$
\int e^{\theta}\sin\theta\ d\theta = \left[\begin{array}{ll} u = \sin\theta, & du = \cos\theta\ d\theta \\ dv = e^{\theta}\ d\theta, & v = e^{\theta} \end{array}\right]
$$

$$
= e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\ d\theta = \left[\begin{array}{ll} u = \cos\theta, & du = -\sin\theta\ d\theta \\ dv = e^{\theta}\ d\theta, & v = e^{\theta} \end{array}\right]
$$

$$
= e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta\ d\theta.
$$

This is the same integral. Therefore

$$
2\int e^{\theta}\sin\theta\ d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta + C,
$$

$$
\int e^{\theta}\sin\theta\ d\theta = \frac{1}{2}\left[e^{\theta}\sin\theta - e^{\theta}\cos\theta\right] + C.
$$

So,

$$
S = 2\sqrt{5}\pi\frac{1}{2}\left[e^{\theta}\sin\theta - e^{\theta}\cos\theta\right]\Big|_0^{\pi/2} = \sqrt{5}\pi\left[e^{\pi/2} + 1\right].
$$

Problems. February 12.

1. Sec. 12.1: # 12. Give a geometric description of the set of points in space whose coordinates satisfy the given pair of equations. $x^2 + (y-1)^2 + z^2 = 4$, $y = 0$.

   Solution.

   Substituting $y = 0$ into $x^2 + (y-1)^2 + z^2 = 4$, we get

   $$x^2 + 1 + z^2 = 4$$

$$x^2 + z^2 = 3.$$

So, this is a circle in the plane $y = 0$, whose center is at the origin and radius is $\sqrt{3}$.

2. Sec. 12.1: # 16 Describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

   a) $x^2 + y^2 \leq 1$, $z = 0$

   Solution. This is the disk of radius 1 centered at the origin that lies in the coordinate plane $z = 0$.

   b) $x^2 + y^2 \leq 1$, $z = 3$

   Solution. This is the disk of radius 1 centered at the point $(0, 0, 3)$ that lies in the coordinate plane $z = 0$.

   c) $x^2 + y^2 \leq 1$, no restriction on z.

   Solution. For each fixed z, we get a disk of radius 1 with center on the z-axis. Taking all such disks, we get the cylinder whose axis is the z-axis and whose sections by the planes parallel to xy-plane are disks of radius 1.

3. Sec. 12.1: # 30. Write inequalities that describe the solid cube in the first octant bounded by the coordinate plane and the planes $x = 2$, $y = 2$, $z = 2$.

   Solution. $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$.

4. Sec. 12.1: # 32 Write inequalities that describe the upper hemisphere of the sphere of radius 1 centered at the origin.

   Solution. $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

5. Sec. 12.1: # 52. Find the center and radius of the sphere $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$.

   Solution.

$$x^2 + y^2 + z^2 + \frac{2}{3}y - \frac{2}{3}z = 3,$$

$$x^2 + y^2 + \frac{2}{3}y + \frac{1}{9} + z^2 - \frac{2}{3}z + \frac{1}{9} - \frac{1}{9} - \frac{1}{9} = 3,$$

$$x^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \frac{29}{9}.$$

   The center of the sphere is $(0, -1/3, 1/3)$, the radius is $\sqrt{29}/3$.

6. Sec. 12.1: # 56. Show that the point $P(3,1,2)$ is equidistant from the points $A(2,-1,3)$ and $B(4,3,1)$.

   Solution.

$$|AP| = \sqrt{(3-2)^2+(1+1)^2+(2-3)^2} = \sqrt{6}.$$

$$|BP| = \sqrt{(3-4)^2+(1-3)^2+(2-1)^2} = \sqrt{6}.$$

   So, $|AP| = |BP|$.